A MINIMAX PROBLEM OF ONE-TIME CORRECTION WITH MEASUREMENT ERRORS

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The simplest model problem of one-time correction of the motion of a system with one degree of freedom in the presence of measurement errors is considered. The optimal correction problem is formulated and solved in minimax (guaranteeing) form, i. e. with allowance for the worst case permitted by the instantaneous measured data.

Such problems have already been considered from the probabilistic standpoint [1]. Close, but somewhat different, minimal problems are investigated in recent paper [2].

1. Formulation of the problem. Let a system with one degree of freedom move uniformly in the absence of control, and let control be effected by a single pulse which can be applied at any instant of motion. The motion of the system is then described by Eqs.

$$\begin{aligned} x &= x_0 + v_0 t, \quad v = v_0 \qquad (0 \leqslant t \leqslant \tau) \\ x &= x_0 + v_0 t + u\,(t - \tau), \quad v = v_0 + u \qquad (\tau < t \leqslant T) \end{aligned} \tag{1.1}$$

Here t is the time, x is the coordinate of the system, v is its velocity, x_0 and v_0 are the values of the coordinate and velocity at the instant $t = 0$, T is the instant of termination of the process, τ is the instant of correction, and u is the magnitude of the correction pulse. The control parameters τ and u are subject to the restrictions

$$0 \leqslant \tau \leqslant T, \qquad |u| \leqslant U \tag{1.2}$$

Here U is the maximum absolute value of the correction pulse. The initial values of the coordinate and velocity are known to within certain errors, i. e. they are subject to the restrictions

$$|x_0 - a| \leqslant \varepsilon, \qquad |v_0 - b| \leqslant \delta \tag{1.3}$$

Here a and b are known approximate values of the coordinate and velocity; ε and δ are the specified measurement errors. The coordinate and velocity of the system are measured continuously during its motion, and inequalities

$$|x\,(t) - y\,(t)| \leqslant \varepsilon, \qquad |v\,(t) - z\,(t)| \leqslant \delta \qquad (0 \leqslant t \leqslant T) \tag{1.4}$$

hold.

Here $y\,(t)$ and $z\,(t)$ are the approximate measured values of the coordinate and velocity at the instant t. It is clear that $y\,(0) = a$, $z\,(0) = b$.

We are required to choose the instant of correction τ and the correction pulse u subject to restrictions (1.2) in such a way as to minimize the deviation (miss) at the end of the process, i.e. the quantity $|x(T)|$. We assume that the measured values of $y(t)$ and $z(t)$ become known immediately, so that the control is determined in accordance with the arriving information. We shall consider the problem in minimax formulation, i.e. with allowance for the worst case permitted by the measured data. The results obtained are therefore the most reliable, i.e. guaranteeing, results.

Determining the optimal control means finding the rule from which it is possible at each instant to find the parameters τ and u on the basis of the measured data up to that instant. Let t_0 be some instant prior to correction. The minimax-optimal values of τ and u corresponding to the instant t_0 are then determined by finding the following minimax:

$$J(t_0) = \min_{\tau} \max_{y,z} \min_{u} \max_{x_0, v_0} |x(T)|, \quad x(T) = x_0 + v_0 T + u(T - \tau) \quad (1.5)$$

Let us explain this relation. The formula for $x(T)$ follows from (1.1). The first minimum in (1.5) is computed with respect to the τ lying in the interval $[t_0, T]$; the second minimum is computed with respect to the u satisfying the condition $|u| \leqslant U$. The first maximum in (1.5) must be determined with respect to all of the functions $y(t)$ and $z(t)$ defined in the interval $(t_0, \tau]$ such that there exists at least one system trajectory which satisfies restrictions (1.4) for all t in the interval $[0, \tau]$. In other words, the measured data must be *noncontradictory*. The functions $y(t)$ and $z(t)$ are otherwise arbitrary. The second maximum in (1.5) must be computed with respect to all the initial values x_0 and v_0 permitted by the measured data prior to the instant τ. These measured data can be broken down into two groups: those obtained for $0 \leqslant t \leqslant t_0$, and those obtained for $t_0 < t \leqslant \tau$. We denote by D_0 the set of points in the parameter plane x_0, v_0, bounded by inequalities

$$|x_0 + v_0 t - y(t)| \leqslant \varepsilon, \qquad |v_0 - z(t)| \leqslant \delta \quad (1.6)$$

where t runs through the values from 0 to t_0. Inequalities (1.6) follow from (1.1) and (1.4). Similarly, D_1 denotes the set of points in the plane x_0, v_0 bounded by inequalities (1.6) for all t from the interval $(t_0, \tau]$. It is clear that the set D_0 can already be determined at the instant $t = t_0$, and that the set D_1 depends on the functions $y(t)$ and $z(t)$ in the interval $(t_0, \tau]$.

The second maximum in relation (1.5) must be determined with respect to all the x_0, and v_0 from the set $D = D_0 \cap D_1$, which is the intersection of the sets D_0 and D_1. The requirement concerning the noncontradictory character of the measured data imposed above on the functions $y(t)$ and $z(t)$ can be formulated as follows: the set D must be nonempty. We note that the sets D_0, D_1 and D are closed convex sets. We have thus characterized completely the domains in which the extrema in (1.5) are computed. The order of succession of the extrema in (1.5) is determined by the order of arrival of the information and of the decisions.

In principle, the optimal control algorithm reduces to the following. New measured data $y(t_0)$ and $z(t_0)$ become known at each instant t_0. This enables us to construct the domain D_0 and to compute minimax (1.5) at this instant. At the same time we determine the quantity $\tau(t_0)$ lying in the interval $[t_0, T]$. If it turns out that $\tau > t_0$, then no correction is required at the instant t_0 and we proceed with tracking, continuously computing the instantaneous values of $\tau(t_0)$. As soon as $\tau(t_0) = t_0$ we make a correction. Here the first maximum in (1.5) can be omitted, since the interval $[t_0, \tau]$ contracts to a point. The magnitude u of the correction pulse is determined on computing the last (in order of recording) two extrema in (1.5), when the domain D coincides with D_0. For $t \geqslant \tau$ the motion is uncontrolled.

The above approach can be extended to more complex (e.g. to many-dimensional) problems of correction with incomplete measured data. However, realization of the optimal control requires determination of minimax (1.5) at each instant even in the simple problem under consideration, and this requirement entails time-consuming computations.

An alternative formulation of the problem is possible. Let the procurement or processing of the tracking data require the time $t_* < T$. The motion is then uncontrolled for $t < t_*$. At any instant $t = t_0 \geqslant t_*$ the instant of correction $\tau(t_0)$ must be determined by computing minimax (1.5), the only differences being that the domain D_0 is determined by inequalities (1.6) for $0 \leqslant t \leqslant t_0 - t_*$ and the domain D_1 by inequalities (1.6) for $t_0 - t_* < t \leqslant \tau - t_*$, and that the functions $y(t)$ and $z(t)$ are considered in the interval $(t_0 - t_*, \tau - t_*]$. In other respects the optimal correction computing scheme remains unchanged.

2. Some simplifications. Minimax (1.5) can be simplified. We note that the functions $y(t)$ and $z(t)$ from which the first maximum in (1.5) is computed affect only the domain D_1 defined by inequalities (1.6) for $t_0 < t \leqslant \tau$. Let us show that these functions can be taken in the form

$$y(t) = y(t_0) + \frac{Y - y(t_0)}{\tau - t_0}(t - t_0), \quad z(t) = z(t_0) \;\Big|\; (t_0 < t \leqslant \tau) \tag{2.1}$$

where Y is some constant.

Let us construct the domain D_1 corresponding to functions (2.1). Substituting (2.1) into the left sides of inequalities (1.6) for $t_0 \leqslant t \leqslant \tau$, we obtain

$$\begin{gathered} |x_0 + v_0 t - y(t)| = (\tau - t_0)^{-1} |x_0(\tau - t_0) + v_0 t(\tau - t_0) - \\ - y(t_0)(\tau - t_0) - [Y - y(t_0)](t - t_0)| = \frac{\tau - t}{\tau - t_0}|x_0 + v_0 t_0 - y(t_0)| + \\ + \frac{t - t_0}{\tau - t_0}|x_0 + v_0\tau - Y|, \quad |v_0 - z(t)| = |v_0 - z(t_0)| \end{gathered} \tag{2.2}$$

Setting $t = t_0$ and $t = \tau$, we obtain from (2.2) and (1.6) inequalities

$$|x_0 + v_0 t_0 - y(t_0)| \leqslant \varepsilon, \quad |v_0 - z(t_0)| \leqslant \delta, \quad |x_0 + v_0\tau - Y| \leqslant \varepsilon \tag{2.3}$$

Inequalities (2.3) belong to the set of inequalities defining the domain D_1. It is easy to verify with the aid of relations (2.2) that if inequalities (2.3) are fulfilled, then inequalities (1.6) are also fulfilled for all t from the interval $[t_0, \tau]$. Thus, inequalities (2.3) define completely the domain D_1 for functions (2.1). But the first two inequalities of (2.3) coincide with (1.6) for $t = t_0$, i.e. they belong to the set of inequalities which define the domain D_0. Since the domain D is the intersection of the domains D_0 and D_1, the first two inequalities of (2.3) can be omitted, since this omission can have no effect on the domains D. Thus, as our domain D_1 corresponding to functions (2.1) we can take the domain of the plane x_0, v_0, defined by inequality

$$|x_0 + v_0\tau - Y| \leqslant \varepsilon \tag{2.4}$$

Let $y(t)$ and $z(t)$ be any pair of functions defined in the interval $(t_0, \tau]$, and let $y(\tau) = Y$. The set of inequalities of the form (1.6) defining the domain D_1 for these functions must then include (for $t = \tau$) inequality (2.4). Hence, if we replace the given functions $y(t)$ and $z(t)$ by functions of the form (2.1), the domain D_1 will not become smaller in any case. This means that the domain D also does not become smaller, and that the maximum over this domain can only increase. Hence, in seeking the maximum in (1.5) we can confine ourselves to functions of the form (2.1) and seek the maximum with respect to the constants Y

Relation (1.5) can be rewritten as

$$J(t_0) = \min_Y \max_u \min_{x_0, v_0} \max |x_0 + v_0 T + u(T - \tau)| \qquad (2.5)$$

The domain D over which we seek the last minimum in (2.5) is the intersection of the domains D_0 and D_1 defined by inequality (2.4). The quantity Y is restricted by the condition that the domain D must be nonempty. The remaining ranges of the parameters involved in computing (2.5) are the same as for (1.5).

The quantity $J(t_0)$ in (2.5) does not increase with increasing t_0 for any measured data. This is because J is a guaranteed estimate of the quantity $|x(T)|$ with allowance for the worst possible future data, so that J can only diminish. The functional $J(t_0)$ has its maximum value $J(0) = J_0$ at the initial instant, when only measured initial data (1.3) are known. The quantity J_0 is an estimate of the miss $|x(T)|$ obtained with the best control and the worst measured data. The possibility of subsequent measurements is taken into account in computing J_0, even though the results of these measurements are not known in advance.

3. Computing the functional. Let us determine the quantities $J_0 = J(0)$, $\tau = \tau(0)$ and u corresponding to the instant $t_0 = 0$ in relation (2.5). The domain D_0 for $t_0 = 0$ is defined by the two inequalities of (1.3). These inequalities imply that

$$a - \varepsilon + (b - \delta)\tau \leqslant x_0 + v_0\tau \leqslant a + \varepsilon + (b + \delta)\tau \qquad (3.1)$$

On the other hand, from inequality (2.4) defining the domain D_1 we obtain

$$Y - \varepsilon \leqslant x_0 + v_0\tau \leqslant Y + \varepsilon \qquad (3.2)$$

We introduce the notation

$$Y' = a + (b - \delta)\tau, \quad Y'' = a + (b + \delta)\tau \qquad (3.3)$$

Comparing relations (3.1) and (3.2), we note that for $Y < Y'$ the left-hand inequality of (3.1) implies fulfillment of the left-hand inequality of (3.2), while for $Y > Y''$ the right-hand inequality of (3.1) implies fulfillment of the right-hand inequality of (3.2). Hence, for $Y < Y'$ or for $Y > Y''$ the domain D is certainly smaller than for $Y = Y'$ or for $Y = Y''$, respectively. Since we seek the maximum over Y in (2.5), and since this maximum does not increase with decreases in the domain D, which depends on Y, it follows that the maximum in (2.5) can be sought only with respect to the Y lying within the bounds $\quad Y' \leqslant Y \leqslant Y'' \qquad (3.4)$

The domain D defined by inequalities (1.3) and (3.2) can be specified in the form

$$a - \varepsilon \leqslant x_0 \leqslant a + \varepsilon, \qquad v' \leqslant v_0 \leqslant v''$$

$$v' = \max[b - \delta, (Y - \varepsilon - x_0)/\tau] v'' = \min[b + \delta, (Y + \varepsilon - x_0)/\tau] \qquad (3.5)$$

It is not difficult to verify that upon fulfillment of conditions (3.4) we have $v' \leqslant v''$ for any x_0 from the interval defined by inequalities (3.5). Hence, the domain D is nonempty under conditions (3.4). Let us rewrite (2.5), omitting the absolute value symbol and computing the maximum with respect to v_0 under restrictions (3.5),

$$J_0 = \min_\tau \max_Y \min_u \max\{[\max_{x_0, v_0}(x_0 + v_0 T) + u(T - \tau)], \qquad (3.6)$$

$$[\max_{x_0, v_0}(-x_0-v_0T)-u(T-\tau)]\} =$$

$$= \min_{\tau}\max_{Y}\min_{u}\max\{[\max_{x_0}(x_0+v''T)+u(T-\tau)],$$

$$[\max_{x_0}(-x_0-v'T)-u(T-\tau)]\}$$

Making use of expression (3.5) for v'', we can write

$$\max_{x_0}(x_0+v''T) = \max_{x_0}\min[f_1(x_0), f_2(x_0)], \qquad a-\varepsilon \leqslant x_0 \leqslant a+\varepsilon$$

$$f_1(x_0) = x_0+(b+\delta)T, \quad f_2(x_0) = [(Y+\varepsilon)T - x_0(T-\tau)]/\tau \quad (3.7)$$

The functions $f_1(x_0)$ and $f_2(x_0)$ are linear in x_0; the former increases and the second decreases with increasing x_0. Hence, maximum (3.7) is attained when $f_1(x_0) = f_2(x_0)$ if the root of this equation, which is given by

$$x'' = Y+\varepsilon-(b+\delta)\tau \quad (3.8)$$

lies in the interval $[a-\varepsilon, a+\varepsilon]$. Inequality (3.4) implies that $x'' \leqslant a+\varepsilon$ at all times. But if $x'' < u-\varepsilon$, then $f_2(x_0) < f_1(x_0)$ for all x_0 from the interval $[a-\varepsilon, a+\varepsilon]$, in which case maximum (3.7) is attained for $x_0 = a-\varepsilon$ and is equal to $f_2(a-\varepsilon)$. Thus, maximum (3.7) is attained for $x_0 = \max(x'', a-\varepsilon)$ and is equal to $F'' = \max_{x_0}(x_0+v''T) = f_2[\max(x'', a-\varepsilon)]$

Taking account of relations (3.7) and (3.8), we can write the above equation in expanded form,

$$F''(Y,\tau) = [(Y-a+2\varepsilon)T+(a-\varepsilon)\tau]/\tau, \qquad Y \leqslant Y_2$$

$$F''(Y,\tau) = Y+\varepsilon+(b+\delta)(T-\tau), \quad Y > Y_2 \quad (3.9)$$

$$Y_2 = a+(b+\delta)\tau-2\varepsilon$$

Relation (3.9) becomes

$$J_0 = \min_{\tau}\max_{Y}\min_{u}\max[F''(Y,\tau)+u(T-\tau), F'(Y,\tau)-u(T-\tau)] \quad (3.10)$$

Here the function F' is computed similarly to F'' in (3.9) and is given by

$$F'(Y,\tau) = -Y+\varepsilon-(b-\delta)(T-\tau), \quad Y \leqslant Y_1$$

$$F'(Y,\tau) = [(-Y+a+2\varepsilon)T-(a+\varepsilon)\tau]/\tau, \quad Y > Y_1$$

$$Y_1 = a+(b-\delta)\tau+2\varepsilon \quad (3.11)$$

The minimum with respect to u for $|u| \leqslant U$ in (3.10) is easy to find, since the minimizable function is the maximum of two functions linear in u. The required minimum is attained either at the point of intersection of the graphs of these functions, or at the bounds of the interval. To put it more precisely, the minimizable value of u is given by

$$u = -U \quad \text{for} \quad F'-F'' \leqslant -2U(T-\tau)$$

$$u = U \quad \text{for} \quad F'-F'' \geqslant 2U(T-\tau)$$

$$u = (F'-F'')/[2(T-\tau)] \quad \text{for} \quad |F'-F''| \leqslant 2U(T-\tau) \quad (3.12)$$

These values of u are associated with the following values of the minimizable function

$$F'' - U(T-\tau), \qquad F' - U(T-\tau), \qquad (F''+F')/2 \tag{3.13}$$

respectively.

It is clear from inequalities (3.12) that the value of u which must be chosen in accordance with these inequalities is always associated with the largest of the three expressions of (3.13). This enables us to rewrite relation (3.10) as

$$J_0 = \min_{\tau} \max_{Y} \max\,[F'' - U(T-\tau),\ F' - U(T-\tau),\ (F'+F'')/2] \tag{3.14}$$

The operation of maximization with respect to Y and the operation of choosing the maximum of the three expressions of (3.14) can be interchanged. According to (3.9) the function F'' increases monotonically with increasing Y; according to (3.11) the function F' decreases monotonically with increasing Y. This means that they attain their maxima at the bounds of interval (3.4). Hence, we can write

$$J_0 = \min\max\,(F_1, F_2, F_3),\ F_1 = F'(Y', \tau) - U(T-\tau)$$

$$F_2 = F''(Y'', \tau) - U(T-\tau),\ F_3 = \max_{Y' \leqslant Y \leqslant Y''} [F'(Y, \tau) + F''(Y,\tau)]/2 \tag{3.15}$$

The relations of (3.3), (3.9), and (3.11) for Y', Y'', Y_1 and Y_2 imply the inequalities $Y' \leqslant Y_1$, $Y'' \geqslant Y_2$. Let us compute the functions F_1, F_2 with allowance for these inequalities and make use of relations (3.15), (3.9), (3.11) and (3.3).

$$F_1 = -Y' + \varepsilon - (b-\delta)(T-\tau) - U(T-\tau) = \\ = -(a+bT) + \varepsilon + \delta T - U(T-\tau)$$

$$F_2 = Y'' + \varepsilon + (b+\delta)(T-\tau) - U(T-\tau) = \\ = a + bT + \varepsilon + \delta T - U(T-\tau)$$

Substituting the above into (3.15), we can rewrite the latter as

$$J_0 = \min\max\ [F_0(\tau), F_3(\tau)]$$

$$F_0(\tau) = |a+bT| + \varepsilon + \delta T - U(T-\tau)$$

$$F_3(\tau) = \max_{Y' \leqslant Y \leqslant Y''} F_4(Y, \tau), \qquad F_4(Y, \tau) = [F'(Y, \tau) + F''(Y, \tau)]/2 \tag{3.16}$$

We consider three cases in computing the function F_3.

1. Let $\delta\tau \leqslant \varepsilon$. Eqs. (3.3), (3.9) and (3.11) here yield

$$Y_2 \leqslant Y' \leqslant Y'' \leqslant Y_1 \tag{3.17}$$

We can compute the functions F_4 and F_3 from relations (3.16), (3.9), (3.11) and condition (3.17),

$$F_4(Y, \tau) = \varepsilon + \delta(T-\tau), \qquad F_3(\tau) = \varepsilon + \delta(T-\tau) \tag{3.18}$$

The function F_4 does not depend on Y in this case.

2. If $\varepsilon \leqslant \delta\tau \leqslant 2\varepsilon$, then from (3.3), (3.9) and (3.11) we find that

$$Y' \leqslant Y_2 \leqslant Y_1 \leqslant Y'' \tag{3.19}$$

Eqs. (3.16), (3.9) and (3.11) under conditions (3.19) imply that the function F_4 is piecewise-linear in Y. As Y increases, the function F_4 increases in the interval $[Y', Y_2]$, remains constant in the interval $[Y_2, Y_1]$, and decreases in the interval $[Y_1, Y'']$. Hence, it attains its maximum, for example, when $Y = Y_1$.

Computing $F_3(\tau) = F_4(Y_1, \tau)$ by means of relations (3.9), (3.11) and inequalities

(3.19), we arrive at our previous formula for F_3 (Formula (3.18)).

3. According to (3.3), (3.9) and (3.11), for $\delta\tau \geqslant 2\varepsilon$ we have

$$Y' \leqslant Y_1 \leqslant Y_2 \leqslant Y'' \tag{3.20}$$

Under conditions (3.20) inequalities (3.16), (3.9) and (3.11) imply that as Y increases, the function F_4 increases in the interval $[Y', Y_1]$, remains constant in the interval $[Y_1, Y_2]$, and decreases in the interval $[Y_2, Y'']$. Its maximum, which it attains when $Y = Y_1$, can be found from relations (3.9), (3.11), (3.16) and (3.20),

$$F_3(\tau) = F_4(Y_1, \tau) = [F'(Y_1, \tau) + F''(Y_1, \tau)] / 2 = \varepsilon(2T - \tau) / \tau \tag{3.21}$$

Recalling Eqs. (3.18) and (3.21), we find that in all of the cases considered

$$F_3(\tau) = \varepsilon + \delta(T - \tau) \text{ for } \tau \leqslant 2\varepsilon\delta^{-1}, \quad F_3(\tau) = \varepsilon(2T - \tau) / \tau \text{ for } \tau \geqslant 2\varepsilon\delta^{-1} \tag{3.22}$$

The function $\max [F_0(\tau), F_3(\tau)]$ defined by Eqs. (3.16) and (3.22) characterizes the minimax value of the miss as a function of the instant of correction τ. Let us now find its minimum with respect to τ for $0 \leqslant \tau \leqslant T$. First, we note that the functions $F_0(\tau)$ and $F_3(\tau)$ defined by Eqs. (3.16) and (3.22) for all real τ are continuous and monotonic, and that $F_0(\tau)$ increases strictly and $F_3(\tau)$ decreases strictly for all τ. Hence, the absolute minimum with respect to τ of the function $\max(F_0, F_3)$ is attained when $F_0(\tau) = F_3(\tau)$.

This equation clearly has a single root τ_* which is easy to find from Eqs. (3.16) and (3.22). Carrying out the necessary computations, we obtain

$$\begin{aligned} \tau_* &= t_1 = (UT - |a + bT|)(U + \delta)^{-1} \quad \text{for } t_1 \leqslant 2\varepsilon\delta^{-1} \\ \tau_* &= t_2 = {}^1/_2\, U^{-1} \{[(|a + bT| + 2\varepsilon + \delta T - UT)^2 + \\ &+ 8U\varepsilon T]^{1/2} + UT - |a + bT| - 2\varepsilon - \delta T\} \quad \text{for } t_1 \geqslant 2\varepsilon\delta^{-1} \end{aligned} \tag{3.23}$$

The two cases, of which only one is realized in any case, correspond to the two analytic expressions for the function F_3 in relations (3.22).

The function $\max(F_0, F_3)$ decreases monotonically for $\tau \leqslant \tau_*$ and increases monotonically for $\tau \geqslant \tau_*$. We can verify with the aid of Eqs. (3.23) that $t_1 \leqslant T$ and $t_2 \leqslant T$, so that $\tau_* \leqslant T$ in all cases. Hence, the minimum in (3.16) with respect to τ from the interval $[0, T]$ is attained either for $\tau = 0$ or for $\tau = \tau_*$. The first case applies if $F_0(0) \geqslant F_3(0)$, or, which is the same thing, if $t_1 \leqslant 0$. The second case is realized for $t_1 \geqslant 0$ and in turn breaks down into two cases in accordance with relations (3.23). The functional in all of the cases can be computed from Formula $J_0 = F_0(\tau)$. Thus, we have

$$\begin{aligned} &\tau = 0, && J_0 = |a + bT| + \varepsilon + \delta T - UT && \text{for } t_1 \leqslant 0 \\ &\tau = t_1, && J_0 = |a + bT| + \varepsilon + \delta T - U(T - t_1) && \text{for } 0 \leqslant t_1 \leqslant 2\varepsilon\delta^{-1} \\ &\tau = t_2, && J_0 = |a + bT| + \varepsilon + \delta T - U(T - t_2) && \text{for } t_1 \geqslant 2\varepsilon\delta^{-1} \end{aligned} \tag{3.24}$$

Substituting the expressions for t_1 and t_2 from (3.23) into (3.24) and carrying out the necessary transformations, we obtain

$$J_0 = |a + bT| + \varepsilon + \delta T - UT, \quad \tau = 0 \quad (UT \leqslant |a + bT|) \tag{3.25}$$

$$J_0 = \varepsilon + \frac{\delta(|a + bT| + \delta T)}{U + \delta}, \quad \tau = \frac{UT - |a + bT|}{U + \delta}$$

$$(UT \geqslant |a + bT|, \quad U(T - 2\varepsilon\delta^{-1}) \leqslant |a + bT| + 2\varepsilon)$$

$$J_0 = {}^1/_2 \{[(|a+bT|+2\varepsilon+\delta T-UT)^2+8U\varepsilon T]^{1/2}+|a+bT|+\delta T-UT\}$$

$$\tau = {}^1/_2 U^{-1}\{[(|a+bT|+2\varepsilon+\delta T-UT)^2+8U\varepsilon T]^{1/2}+UT-|a+bT|-2\varepsilon-\delta T\}$$

$$(U(T-2\varepsilon\delta^{-1})\geqslant|a+bT|+2\varepsilon)$$

Formulas (3.25) define the required quantity $J_0' = J(0)$ in all cases. We note that result (3.25) remains unchanged if we assume that only the coordinate is measured for $t > 0$. This is because we derived (3.25) assuming the worst measured results, the worst measured data for the velocity being $z(t) = b$ for $t > 0$. These measurements clearly add nothing to known initial measured data (1.3).

If we assume that measurement of the coordinate also ceases for $t > 0$ result (3.25) changes. In this case the control is based solely on initial measured data (1.3). The minimax value of the functional is then given by

$$J_1 = \min_{\tau}\min_{u}\max_{x_0, v_0}|x_0+v_0T+u(T-\tau)| \qquad (3.26)$$

where the maximum is computed over the rectangle defined by inequalities (1.3). Minimax (3.26) can be computed similarly to, much more simply than, (2.5).

Determining the maximum, we obtain the following expression in place of (3.10):

$$J_1 = \min_{\tau}\min_{u}\max[F''+u(T-\tau),\ F'-u(T-\tau)]$$

$$F'' = a+\varepsilon+(b+\delta)T, \qquad F' = -a+\varepsilon-(b-\delta)T \qquad (3.27)$$

Here the quantities F' and F'' are independent of Y and τ. Computing the minimum with respect to u, we find as with (3.16) that

$$J_1 = \min_{\tau}\max[|a+bT|+\varepsilon+\delta T-U(T-\tau),\ \varepsilon+\delta T]$$

The function whose minimum we are required to find does not decrease with increasing τ. The minimum is therefore attained for $\tau = 0$. The required functional, the corresponding instant of correction, and the magnitude of the pulse in this case are

$$J_1 = |a+bT|+\varepsilon+\delta T-UT, \qquad \tau = 0, \qquad u = -U\operatorname{sgn}(a+bT)$$

for $UT \leqslant |a+bT|$ and

$$J_1 = \varepsilon+\delta T, \qquad \tau = 0, \qquad u = -(a+bT)/T \qquad (3.28)$$

for $UT \geqslant |a+bT|$.

The control u is here computed from earlier Formulas (3.12) with allowance for the values of F' and F'' from (3.27) and for the equality $\tau = 0$. Comparing (3.25) and (3.28), we see that $J_0 = J_1$ only when $\tau = 0$ in (3.25). In the remaining cases, i.e. when $\tau > 0$ in relations (3.25), we have $J_0 \leqslant J_1$, which follows from the formulation of the problem, but can also be verified directly.

Let us consider Formulas (3.25) in the case $U \to \infty$. This happens when the possible correction pulse is large as compared with the errors in the initial conditions and measurements. Relations (3.25) then yield

$$J_0 = \varepsilon + O(U^{-1}), \quad \tau = T-(|a+bT|+\delta T)U^{-1}+O(U^{-2})$$

for all a, b, ε, δ and T

The miss in this case is close to the error of measuring the coordinate ε and the instant of correction τ is close to the instant T of termination of the process.

Relations (2.5) indicate that only measurement of the coordinate at the assumed instant of correction τ, i.e. of the quantity $Y = y(\tau)$, is essential in solving the minimax problem. Hence, solution of (3.25) can also be regarded as computation of the optimal correction in the case when only one measurement is possible during motion. It is expedient to make this measurement immediately before correction, i.e. at the instant τ given by Eqs. (3.25).

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ON THE SOLUTION OF VARIATIONAL PROBLEMS OF SUPERSONIC FLOWS OF GAS WITH FOREIGN PARTICLES

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Questions arising in solving the problem of design of the optimum contour of the supersonic portion of plane and axisymmetric nozzles for flows involving any nonequilibrium processes are considered. An investigation is carried out on the example of the flow of gas containing foreign particles (solid, or liquid) by using Lagrange multipliers in the form first applied to problems of supersonic gas dynamics by Guderley and Armitage [1].

The exactly formulated problem of design of the supersonic portion of plane and axisymmetric nozzles for nonequilibrium flows were considered in papers [2 and 3], while papers [4 and 5] dealt with the problem of flow of gas with foreign particles. In deriving the conditions necessary for the determination of the optimum authors of these papers had considered that case only in which the first set characteristic bounding on the right the region of influence of the sought contour intersects the rarefaction wave beam closing characteristic originating in the flow past the starting point of a (contour) kink, or in the case of a curvature constraint in the flow past the initial section of the maximum permissible curvature (*). The consideration of that case only appeared natural, as for

*) As will be clear from the following, in this case in the system of conditions derived in [4 and 5] the conditions along the particle streamline separating the region containing particles from the particle-free gas have been lost.